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**A NOTE ON BIVARIATE BI-PERIODIC JACOBSTHAL
POLYNOMIALS AND BIVARIATE BI-PERIODIC JACOBSTHAL-
LUCAS POLYNOMIALS**

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ABSTRACT

In this paper we introduce two polynomial sequences first one is Bivariate Bi- periodic Jacobsthal polynomial which is define as $j_n(x, z) = u(x, z)j_{n-1}(x, z) + 2h(z)j_{n-2}(x, z)$ if n is even and $j_n(x, z) = v(x, z)j_{n-1}(x, z) + 2h(z)j_{n-2}(x, z)$ if n is odd with initial conditions $j_0(x, z) = 0$ and $j_1(x, z) = 1$ and second one is Bivariate Bi- periodic Jacobsthal Lucas polynomials which is define as $\hat{c}_n(x, z) = v(x, z)\hat{c}_{n-1}(x, z) + 2h(z)\hat{c}_{n-2}(x, z)$ if n is even and $\hat{c}_n(x, z) = u(x, z)\hat{c}_{n-1}(x, z) + 2h(z)\hat{c}_{n-2}(x, z)$ if n is odd for $n \geq 2$ with initial conditions $\hat{c}_0(x, z) = 2$, $\hat{c}_1(x, z) = u(x, z)$. We have found generating function and Binet's formula of both the polynomial sequences. Investigate relationship between Bivariate Bi- periodic Jacobsthal and Bivariate Bi- periodic Jacobsthal Lucas polynomials. Also find well-known Cassini's identity, Catalan's identity and discuss the converging properties of both the polynomial sequences.

Keywords: Bi-variate Bi-periodic Jacobsthal polynomials, Bi-variate Bi-periodic Jacobsthal Lucas polynomial, Cassini's identity, Catalan's identity, Binet's formula, Generating function

1.INTRODUCTION

In 2002 [2] Catalani introduced the generalisation of bivariate polynomials and some properties by matrix approach (mainly focused on Fibonacci and Lucas polynomials). The contribution of Catalani

is noteworthy on bivariate polynomials [3]. After Catalani many authors worked on bivariate Fibonacci and Lucas polynomials find many properties and relationship between bivariate Fibonacci and Lucas

polynomials. In 2016 Uygun [7] defined generalization of Jacobsthal sequence and proved many identities. In 2018 Uygun [9] defined Jacobsthal-Lucas and bivariate Jacobsthal matrix polynomial sequences investigated important properties, relationship between bivariate Jacobsthal-Lucas and Jacobsthal matrix polynomial sequences. In 2019 Uygun [10, 11] introduced generalized bi-periodic Jacobsthal-Lucas sequences and generalized Jacobsthal-Lucas sequences and obtained some properties.

In 2020 [1, 12] Verma and Bala introduced a new generalisation as bi-periodic Jacobsthal polynomial and bivariate bi-periodic Fibonacci polynomial, prove some identities based on these polynomials.

$$j_n(x, z) = \begin{cases} u(x, z)j_{n-1}(x, z) + 2h(z)j_{n-2}(x, z) & \text{if } n \text{ is even} \\ v(x, z)j_{n-1}(x, z) + 2h(z)j_{n-2}(x, z) & \text{if } n \text{ is odd} \end{cases} \quad n \geq 2 \tag{2.1.1}$$

with initial conditions $j_0(x, z) = 0, j_1(x, z) = 1$

$$j_0(x, z) = 0, \quad j_1(x, z) = 1, \quad j_2(x, z) = u(x, z), \quad j_3(x, z) = u(x, z)v(x, z) + 2h(z), \\ j_4(x, z) = u(x, z)^2v(x, z) + 4u(x, z)h(z)$$

Alternative Definition: For any $u(x, z), v(x, z)$ and $h(z)$ belonging to $\mathbb{R} - \{0\}$, x, z belonging to \mathbb{R} and $n \in \mathbb{N}$, the bivariate bi-periodic Jacobsthal polynomials is defined by

$$j_n(x, z) = u(x, z)^{1-\xi(n)}v(x, z)^{\xi(n)}j_{n-1}(x, z) + 2h(z)j_{n-2}(x, z), \quad n \geq 2 \tag{2.1.2}$$

with $j_0(x, z) = 0, j_1(x, z) = 1$

where $\xi(n) = n - 2 \lfloor \frac{n}{2} \rfloor$

Definition 2.2: For any $u(x, z), v(x, z)$ and $h(z)$ belonging to $\mathbb{R} - \{0\}$, x, z belonging to \mathbb{R} and $n \in \mathbb{N}$, the bivariate bi-periodic Jacobsthal Lucas polynomials is defined by

$$\hat{c}_n(x, z) = \begin{cases} v(x, z)\hat{c}_{n-1}(x, z) + 2h(z)\hat{c}_{n-2}(x, z) & \text{if } n \text{ is even} \\ u(x, z)\hat{c}_{n-1}(x, z) + 2h(z)\hat{c}_{n-2}(x, z) & \text{if } n \text{ is odd} \end{cases} \quad n \geq 2, \tag{2.1.3}$$

with initial conditions are $\hat{c}_0(x, z) = 2, \hat{c}_1(x, z) = u(x, z)$

Alternative Definition: For any $u(x, z), v(x, z)$ and $h(z)$ belonging to $\mathbb{R} - \{0\}$, x, z belonging to \mathbb{R} and $n \in \mathbb{N}$, the bivariate bi-periodic Jacobsthal Lucas polynomials is defined by

Here, firstly we introduce the bivariate bi-periodic Jacobsthal and bivariate bi-periodic Jacobsthal-Lucas polynomial sequences. Find Binet’s formula and generating function for both polynomial sequences. Investigate relationship between bivariate bi-periodic Jacobsthal and bivariate bi-periodic Jacobsthal Lucas polynomials. Prove some well-known Cassini’s identity, Catalan’s identity and discuss the converging properties of both the polynomial sequences.

2. Definitions and results

Definition 2.1: For any $u(x, z), v(x, z)$ and $h(z)$ belonging to $\mathbb{R} - \{0\}$, x, z belonging to \mathbb{R} and $n \in \mathbb{N}$ the bivariate bi-periodic Jacobsthal polynomials is given by:

$$\hat{c}_n(x, z) = v(x, z)^{1-\xi(n)}u(x, z)^{\xi(n)}\hat{c}_{n-1}(x, z) + 2h(z)\hat{c}_{n-2}(x, z), \quad n \geq 2 \tag{2.1.4}$$

with $\hat{c}_0(x, z) = 2, \hat{c}_1(x, z) = u(x, z)$

where $\xi(n) = n - 2 \lfloor \frac{n}{2} \rfloor$

From the definition (2.1.1) and (2.1.3), characteristic equation of the bivariate bi-periodic Jacobsthal polynomial and bivariate bi-periodic Jacobsthal Lucas polynomials are

$$\lambda^2 - u(x, z)v(x, z)\lambda - 2u(x, z)v(x, z)h(z) = 0$$

with roots $\lambda_1(x, z)$ and $\lambda_2(x, z)$ given by

$$\lambda_1(x, z) = \frac{u(x, z)v(x, z) + \sqrt{(u(x, z)v(x, z))^2 + 8u(x, z)v(x, z)h(z)}}{2}$$

and

$$\lambda_2(x, z) = \frac{u(x, z)v(x, z) - \sqrt{(u(x, z)v(x, z))^2 + 8u(x, z)v(x, z)h(z)}}{2}$$

Lemma 2.1 If $\{\hat{c}_n(x, z)\}_{n=0}^\infty$ and $\{j_n(x, z)\}_{n=0}^\infty$ are given by (2.1.1) and (2.1.3) then:

$$j_{2n}(x, z) = (u(x, z)v(x, z) + 4h(z))j_{2n-2}(x, z) - (2h(z))^2j_{2n-4}(x, z) \tag{2.1.5}$$

$$j_{2n+1}(x, z) = (u(x, z)v(x, z) + 4h(z))j_{2n-1}(x, z) - (2h(z))^2j_{2n-3}(x, z) \tag{2.1.6}$$

$$\hat{c}_{2n}(x, z) = (u(x, z)v(x, z) + 4h(z))\hat{c}_{2n-2}(x, z) - (2h(z))^2\hat{c}_{2n-4}(x, z) \tag{2.1.7}$$

$$\hat{c}_{2n+1}(x, z) = (u(x, z)v(x, z) + 4h(z))\hat{c}_{2n-1}(x, z) - (2h(z))^2\hat{c}_{2n-3}(x, z) \tag{2.1.8}$$

Proof: By using definition 2.1

$$\begin{aligned} j_{2n}(x, z) &= u(x, z)j_{2n-1}(x, z) + 2h(z)j_{2n-2}(x, z) \\ &= u(x, z)[v(x, z)j_{2n-2}(x, z) + 2h(z)j_{2n-3}(x, z)] + 2h(z)j_{2n-2}(x, z) \\ &= (u(x, z)v(x, z) + 2h(z))j_{2n-2}(x, z) + 2h(z)[u(x, z)j_{2n-3}(x, z)] \\ &= (u(x, z)v(x, z) + 2h(z))j_{2n-2}(x, z) + 2h(z)[j_{2n-2}(x, z) - 2h(z)j_{2n-4}(x, z)] \\ &= (u(x, z)v(x, z) + 4h(z))j_{2n-2}(x, z) - (2h(z))^2j_{2n-4}(x, z) \end{aligned}$$

Similarly, we can prove

$$\begin{aligned} j_{2n+1}(x, z) &= (u(x, z)v(x, z) + 4h(z))j_{2n-1}(x, z) - (2h(z))^2j_{2n-3}(x, z) \\ \hat{c}_{2n}(x, z) &= (u(x, z)v(x, z) + 4h(z))\hat{c}_{2n-2}(x, z) - (2h(z))^2\hat{c}_{2n-4}(x, z) \\ \hat{c}_{2n+1}(x, z) &= (u(x, z)v(x, z) + 4h(z))\hat{c}_{2n-1}(x, z) - (2h(z))^2\hat{c}_{2n-3}(x, z) \end{aligned}$$

Lemma 2.2 For $n \in \mathbb{N}$, if $\{j_n(x, z)\}_{n=0}^\infty$ is defined by (2.1.1) then,

$$j_{n+6}(x, z) = (u(x, z)v(x, z) + 4h(z))u(x, z)^{1-\xi(n)}v(x, z)^{\xi(n)}j_{n+3}(x, z) - 8(h(z))^2j_n(x, z)$$

Proof: This can be easily proven with the help of alternate definition of bivariate bi-periodic Jacobsthal polynomial.

Lemma 2.3 The $\lambda_1(x, z)$ and $\lambda_2(x, z)$ defined by (2.1.2) satisfy the following properties

- (i) $(\lambda_1 + 2h(z))(\lambda_2 + 2h(z)) = (2h(z))^2$
- (ii) $\lambda_1 + \lambda_2 = u(x, z)v(x, z)$
- (iii) $\lambda_1 \lambda_2 = -2u(x, z)v(x, z)h(z)$
- (iv) $(\lambda_1)^2 + (\lambda_2)^2 = (u(x, z)v(x, z))^2 + 4u(x, z)v(x, z)h(z)$
- (v) $-(\lambda_1(\lambda_2 + 2h(z))) = 2h(z)\lambda_2$
- (vi) $-(\lambda_2(\lambda_1 + 2h(z))) = 2h(z)\lambda_1$
- (vii) $(\lambda_2 + 2h(z)) = \frac{(\lambda_2)^2}{u(x, z)v(x, z)}$
- (viii) $(\lambda_1 + 2h(z)) = \frac{(\lambda_1)^2}{u(x, z)v(x, z)}$

We can use $\lambda_1(x, z) = \lambda_1$ and $\lambda_2(x, z) = \lambda_2$

3. Generating function, some identities, binomial summation formula and relationship between $\{j_n(x, z)\}_{n=0}^\infty$ and $\{\hat{c}_n(x, z)\}_{n=0}^\infty$.

Theorem 3.1(Generating Function). For every $m \in \mathbb{N}$, the generating function of the bivariate bi-periodic Jacobsthal and bivariate bi-periodic Jacobsthal Lucas polynomial sequences are denoted by $J(t)$ and $C(t)$ are given by

$$J(t) = \frac{t(1 + u(x, z)t - 2h(z)t^2)}{1 - (u(x, z)v(x, z) + 4h(z))t^2 + (2h(z))^2t^4}$$

$$C(t) = \frac{2 - (u(x, z)v(x, z) + 4h(z))t^2 + u(x, z)t + 2h(z)u(x, z)t^3}{1 - (u(x, z)v(x, z) + 4h(z))t^2 + (2h(z))^2t^4}$$

Proof: $J(t)$ is representing in the form of power series

$$J(t) = \sum_{m=0}^\infty j_m(x, z)t^m$$

We can write $J(t)$ as $J(t) = J_0(t) + J_1(t)$

Where the $J_0(t)$ is even part of the series

$$J_0(t) = j_0(x, z) + j_2(x, z)t^2 + \dots = \sum_{m=0}^\infty j_{2m}(x, z)t^{2m}$$

And $J_1(t)$ is odd part of the series

$$J_1(t) = j_1(x, z)t + j_3(x, z)t^3 + \dots = \sum_{m=0}^\infty j_{2m+1}(x, z)t^{2m+1}$$

Now consider the even part of the series

$$J_0(t) = j_0(x, z) + j_2(x, z)t^2 + \dots = \sum_{m=0}^\infty j_{2m}(x, z)t^{2m} = 0 + u(x, z)t^2 + \sum_{m=2}^\infty j_{2m}(x, z)t^{2m} \quad (3.1.1)$$

by Lemma 2.2

$$j_{2m}(x, z) = (u(x, z)v(x, z) + 4h(z))j_{2m-2}(x, z) - (2h(z))^2j_{2m-4}(x, z)$$

on substituting the above value of $j_{2m}(x, z)$ in (3.1.1)

$$J_0(t) = u(x, z)t^2 + \sum_{m=2}^{\infty} [(u(x, z)v(x, z) + 4h(z))j_{2m-2}(x, z) - (2h(z))^2 j_{2m-4}(x, z)]t^{2m}$$

$$J_0(t) = u(x, z)t^2 + (u(x, z)v(x, z) + 4h(z))t^2 J_0(t) - (2h(z))^2 t^4 J_0(t)$$

Hence, we get

$$J_0(t) = \frac{u(x, z)t^2}{1 - (u(x, z)v(x, z) + 4h(z))t^2 + (2h(z))^2 t^4}$$

Similarly, we can find

$$J_1(t) = \frac{t - 2h(z)t^3}{1 - (u(x, z)v(x, z) + 4h(z))t^2 + (2h(z))^2 t^4}$$

Since

$$J(t) = J_0(t) + J_1(t)$$

$$J(t) = \frac{t(1 + u(x, z)t - 2h(z)t^2)}{1 - (u(x, z)v(x, z) + 4h(z))t^2 + (2h(z))^2 t^4}$$

In the same way we can find the generating function bivariate bi-periodic Jacobsthal Lucas polynomials as

$$C(t) = \frac{2 - (u(x, z)v(x, z) + 4h(z))t^2 + u(x, z)t + 2h(z)u(x, z)t^3}{1 - (u(x, z)v(x, z) + 4h(z))t^2 + (2h(z))^2 t^4}$$

Theorem 3.2 (Binet’s formula) : For every $n \in \mathbb{N}$, n^{th} term of $\{j_n(x, z)\}_{n=0}^{\infty}$ and $\{\hat{c}_n(x, z)\}_{n=0}^{\infty}$ are given by

$$j_n(x, z) = \frac{u(x, z)^{1-\xi(n)}}{(u(x, z)v(x, z))^{\lfloor \frac{n}{2} \rfloor}} \left(\frac{\lambda_1^n - \lambda_2^n}{\lambda_1 - \lambda_2} \right) \tag{3.2.1}$$

$$\hat{c}_n(x, z) = \frac{u(x, z)^{\xi(n)}}{(u(x, z)v(x, z))^{\lfloor \frac{n+1}{2} \rfloor}} (\lambda_1^n + \lambda_2^n) \tag{3.2.2}$$

where the parity function is $\xi(n) = n - 2 \lfloor \frac{n}{2} \rfloor$.

Proof: Parity function $\xi(n)$, can be expressed as

$$\xi(n) = \begin{cases} 0 & \text{if } n \text{ is even} \\ 1 & \text{if } n \text{ is odd} \end{cases}$$

we can find Binet’s formula using two step method

1. Take generating function and using partial fraction decomposition method.
2. After using partial fraction apply Maclaurin’s Series expansion.

we get,

$$J(t) = \frac{1}{(2h(z))^2(\lambda_1 - \lambda_2)} \left(\frac{u(x, z)(\lambda_1 + 2h(z)) - 2h(z)\lambda_1 t}{t^2 - \frac{(\lambda_1 + 2h(z))}{(2h(z))^2}} - \frac{u(x, z)(\lambda_2 + 2h(z)) - 2h(z)\lambda_2 t}{t^2 - \frac{(\lambda_2 + 2h(z))}{(2h(z))^2}} \right)$$

We know that $J(t) = J_0(t) + J_1(t)$

Where $J_0(t)$ is even term series and $J_1(t)$ is odd term series and after separating series, we can use Maclaurin's Series expansion and identities $-(\lambda_1(\lambda_2 + 2h(z)) = 2h(z)\lambda_2$ and $-(\lambda_2(\lambda_1 + 2h(z)) = 2h(z)\lambda_1$ and we get,

$$J_0(t) = \frac{u(x, z)}{(2h(z))^2(\lambda_1 - \lambda_2)} \cdot \sum_{n=0}^{\infty} \left((2h(z))^2 [(\lambda_1 + 2h(z))^n - (\lambda_2 + 2h(z))^n] \right) t^{2n}$$

$$J_1(t) = \frac{1}{(2h(z))^2(\lambda_1 - \lambda_2)} \cdot \sum_{n=0}^{\infty} \left((2h(z))^2 [\lambda_1 \{(\lambda_1 + 2h(z))^n\} - \{\lambda_2(\lambda_2 + 2h(z))^n\}] \right) t^{2n+1}$$

Using identities $(\lambda_2 + 2h(z)) = \frac{(\lambda_2)^2}{u(x,z)v(x,z)}$ and $(\lambda_1 + 2h(z)) = \frac{(\lambda_1)^2}{u(x,z)v(x,z)}$

We find,

$$J_0(t) = \sum_{n=0}^{\infty} \frac{u(x, z)}{(u(x, z)v(x, z))^n(\lambda_1 - \lambda_2)} ([(\lambda_1)^{2n} - (\lambda_2)^{2n}]) t^{2n}$$

$$J_1(t) = \sum_{n=0}^{\infty} \frac{1}{(u(x, z)v(x, z))^n(\lambda_1 - \lambda_2)} ([(\lambda_1)^{2n+1} - (\lambda_2)^{2n+1}]) t^{2n+1}$$

After solving $J_0(t)$ and $J_1(t)$ we get

$$J(t) = \sum_{n=0}^{\infty} \frac{u(x, z)^{1-\xi(n)}}{[u(x, z)v(x, z)]^{\lfloor \frac{n}{2} \rfloor}} \left(\frac{\lambda_1^n - \lambda_2^n}{\lambda_1 - \lambda_2} \right) t^n$$

So, we have Binet's formula

$$j_n(x, z) = \frac{u(x, z)^{1-\xi(n)}}{(u(x, z)v(x, z))^{\lfloor \frac{n}{2} \rfloor}} \left(\frac{\lambda_1^n - \lambda_2^n}{\lambda_1 - \lambda_2} \right)$$

Applying the same processor, we can obtain Binet's formula for finding n^{th} term of $\{\hat{c}_n(x, z)\}_{n=0}^{\infty}$ as

$$\hat{c}_n(x, z) = \frac{u(x, z)^{\xi(n)}}{(u(x, z)v(x, z))^{\lfloor \frac{n+1}{2} \rfloor}} (\lambda_1^n + \lambda_2^n)$$

Theorem 3.3 For every two consecutive terms of the bi-periodic Jacobsthal polynomials

$$\lim_{n \rightarrow \infty} \frac{j_{2n+1}(x, z)}{j_{2n}(x, z)} = \frac{\lambda_1}{u(x, z)} = \lim_{n \rightarrow \infty} \frac{\hat{c}_{2n}(x, z)}{\hat{c}_{2n-1}(x, z)}$$

and

$$\lim_{n \rightarrow \infty} \frac{j_{2n}(x, z)}{j_{2n-1}(x, z)} = \frac{\lambda_1}{v(x, z)} = \lim_{n \rightarrow \infty} \frac{\hat{c}_{2n+1}(x, z)}{\hat{c}_{2n}(x, z)}$$

Proof: Consider

$$|\lambda_2| \leq \lambda_1 \text{ and } \lim_{n \rightarrow \infty} \left(\frac{\lambda_2}{\lambda_1}\right)^n = 0$$

by equation (3.2.1) we have

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{j_{2n+1}(x, z)}{j_{2n}(x, z)} &= \lim_{n \rightarrow \infty} \frac{\frac{1}{(u(x, z)v(x, z))^{\lfloor \frac{2n+1}{2} \rfloor}} (\lambda_1^{2n+1} - \lambda_2^{2n+1})}{\frac{u(x, z)}{(u(x, z)v(x, z))^{\lfloor \frac{2n}{2} \rfloor}} (\lambda_1^{2n} - \lambda_2^{2n})} \\ &= \lim_{n \rightarrow \infty} \frac{(u(x, z)v(x, z))^{\lfloor \frac{2n}{2} \rfloor} (\lambda_1^{2n+1} - \lambda_2^{2n+1})}{u(x, z) \cdot (u(x, z)v(x, z))^{\lfloor \frac{2n+1}{2} \rfloor} (\lambda_1^{2n} - \lambda_2^{2n})} \\ &= \lim_{n \rightarrow \infty} \frac{\lambda_1^{2n+1} \left(1 - \frac{\lambda_2^{2n+1}}{\lambda_1^{2n+1}}\right)}{u(x, z) \cdot \lambda_1^{2n} \left(1 - \frac{\lambda_2^{2n}}{\lambda_1^{2n}}\right)} = \lim_{n \rightarrow \infty} \frac{\lambda_1}{u(x, z)} = \frac{\lambda_1}{u(x, z)} \end{aligned}$$

Again, apply same process and we get

$$\lim_{n \rightarrow \infty} \frac{\hat{c}_{2n}(x, z)}{\hat{c}_{2n-1}(x, z)} = \frac{\lambda_1}{u(x, z)}$$

and

$$\lim_{n \rightarrow \infty} \frac{j_{2n}(x, z)}{j_{2n-1}(x, z)} = \frac{\lambda_1}{v(x, z)} = \lim_{n \rightarrow \infty} \frac{\hat{c}_{2n+1}(x, z)}{\hat{c}_{2n}(x, z)}$$

Theorem 3.4 If $\{j_n(x, z)\}_{n=0}^{\infty}$ is defined by (2.1) then prove that

$$j_{-n}(x, z) = (-1)^{n+1} (2h(z))^{-n} j_n(x, z), \quad n = 1, 2, 3, \dots,$$

Proof: By using Binet’s formula on replacing n by $-n$ we have result.

Theorem 3.5 (Catalan’s Identity): If n and r are positive integers with $n \geq r$, then prove that

$$\begin{aligned} &u(x, z)^{\xi(n-r)} v(x, z)^{1-\xi(n-r)} j_{n-r}(x, z) j_{n+r}(x, z) - u(x, z)^{\xi(n)} v(x, z)^{1-\xi(n)} j_n^2(x, z) \\ &= -(-2h(z))^{n-r} u(x, z)^{\xi(r)} v(x, z)^{1-\xi(r)} j_r^2(x, z) \end{aligned}$$

Proof: By using equation (3.2.1)

$$\begin{aligned} &u(x, z)^{\xi(n-r)} v(x, z)^{1-\xi(n-r)} j_{n-r}(x, z) j_{n+r}(x, z) \\ &= u(x, z)^{\xi(n-r)} v(x, z)^{1-\xi(n-r)} \frac{u(x, z)^{1-\xi(n-r)}}{(u(x, z)v(x, z))^{\lfloor \frac{n-r}{2} \rfloor}} \frac{u(x, z)^{1-\xi(n+r)}}{(u(x, z)v(x, z))^{\lfloor \frac{n+r}{2} \rfloor}} \left(\frac{\lambda_1^{n-r} - \lambda_2^{n-r}}{\lambda_1 - \lambda_2}\right) \left(\frac{\lambda_1^{n+r} - \lambda_2^{n+r}}{\lambda_1 - \lambda_2}\right) \\ &= \frac{u(x, z)^{2-\xi(n+r)} v(x, z)^{1-\xi(n-r)}}{(u(x, z)v(x, z))^{n-\xi(n-r)}} \left(\frac{\lambda_1^{n-r} - \lambda_2^{n-r}}{\lambda_1 - \lambda_2}\right) \left(\frac{\lambda_1^{n+r} - \lambda_2^{n+r}}{\lambda_1 - \lambda_2}\right) \\ &= \frac{u(x, z)}{(u(x, z)v(x, z))^{n-1}} \left[\frac{\lambda_1^{2n} - (\lambda_1 \lambda_2)^{n-r} (\lambda_1^{2r} + \lambda_2^{2r}) + \lambda_2^{2n}}{(\lambda_1 - \lambda_2)^2} \right] \end{aligned}$$

Again, by using equation (3.2.1)

$$u(x, z)^{\xi(n)} v(x, z)^{1-\xi(n)} j_n^2(x, z) = \frac{u(x, z)}{(\lambda_1 - \lambda_2)^2 (u(x, z) v(x, z))^{n-1}} [\lambda_1^{2n} - 2(\lambda_1 \lambda_2)^n + \lambda_2^{2n}]$$

Therefore

$$\begin{aligned} & u(x, z)^{\xi(n-r)} v(x, z)^{1-\xi(n-r)} j_{n-r}(x, z) j_{n+r}(x, z) - u(x, z)^{\xi(n)} v(x, z)^{1-\xi(n)} j_n^2(x, z) \\ &= \frac{u(x, z)}{(u(x, z) v(x, z))^{n-1} ((\lambda_1 - \lambda_2)^2)^2} [-(\lambda_1 \lambda_2)^{n-r} (\lambda_1^{2r} + \lambda_2^{2r}) + 2(\lambda_1 \lambda_2)^n] \\ &= \frac{u(x, z)}{(u(x, z) v(x, z))^{n-1} ((\lambda_1 - \lambda_2)^2)^2} [-(\lambda_1 \lambda_2)^{n-r} [(\lambda_1^{2r} + \lambda_2^{2r}) - 2(\lambda_1 \lambda_2)^r]] \\ &= \frac{-u(x, z) (\lambda_1 \lambda_2)^{n-r} \left[\frac{\lambda_1^r - \lambda_2^r}{\lambda_1 - \lambda_2} \right]^2}{(u(x, z) v(x, z))^{n-1}} \\ &= \frac{-(-2u(x, z) v(x, z) h(z))^{n-r} u(x, z) \left[\frac{\lambda_1^r - \lambda_2^r}{\lambda_1 - \lambda_2} \right]^2}{(u(x, z) v(x, z))^{n-1}} \\ &= \frac{-(-2u(x, z) v(x, z) h(z))^{n-r} u(x, z) (u(x, z) v(x, z))^2 \left[\frac{r}{2} \right]}{(u(x, z) v(x, z))^{n-1} u(x, z)^{2-2\xi(r)}} j_r^2(x, z) \\ &= -(-2h(z))^{n-r} u(x, z)^{\xi(r)} v(x, z)^{1-\xi(r)} j_r^2(x, z) \end{aligned}$$

Theorem 3.6 (Cassini's Identity) For any natural number n , prove that

$$\left(\frac{u(x, z)}{v(x, z)} \right)^{\xi(n-1)-1} j_{n-1}(x, z) j_{n+1}(x, z) - \left(\frac{u(x, z)}{v(x, z)} \right)^{\xi(n)-1} j_n^2(x, z) = -(-2h(z))^{n-1}$$

Proof: Put the value of $r = 1$ in Catalan's identity we get the Cassini's identity.

Theorem 3.7 (D'Ocagne's Identity) For any two positive integers m and n , with $m \geq n$, prove that

$$u(x, z)^{\xi(mn+m)} v(x, z)^{\xi(mn+n)} j_m(x, z) j_{n+1}(x, z) - u(x, z)^{\xi(mn+n)} v(x, z)^{\xi(mn+m)} j_{m+1}(x, z) j_n(x, z) = -(-2h(z))^n u(x, z) j_{m-n}(x, z)$$

Proof: Proof can be easily obtained from Binet's formula (3.2.1) such that

Let us assume

$$u(x, z)^{\xi(mn+m)} v(x, z)^{\xi(mn+n)} j_m(x, z) j_{n+1}(x, z) = \Psi(x, z)$$

and

$$u(x, z)^{\xi(mn+n)} v(x, z)^{\xi(mn+m)} j_{m+1}(x, z) j_n(x, z) = \Phi(x, z)$$

where

$$\Psi(x, z) = \frac{u(x, z) [u(x, z) v(x, z)]^{-n}}{(u(x, z) v(x, z))^{\frac{m-n-\xi(m-n)}{2}}} \left[\frac{\lambda_1^{m+n+1} - (\lambda_1 \lambda_2)^m (\lambda_1 \lambda_2^{m-n} + \lambda_2 \lambda_1^{m-n}) + \lambda_2^{m+n+1}}{(\lambda_1 - \lambda_2)^2} \right]$$

and

$$\Phi(x, z) = \frac{u(x, z) [u(x, z) v(x, z)]^{-n}}{(u(x, z) v(x, z))^{\frac{m-n-\xi(m-n)}{2}}} \left[\frac{\lambda_1^{m+n+1} - (\lambda_1 \lambda_2)^n (\lambda_2^{m-n+1} + \lambda_1^{m-n+1}) + \lambda_2^{m+n+1}}{(\lambda_1 - \lambda_2)^2} \right]$$

Now we subtract $\Psi(x, z)$ and $\Phi(x, z)$ we get,

$$\begin{aligned} \forall(x, z) - \epsilon(x, z) &= \frac{u(x, z)[u(x, z)v(x, z)]^{-n}}{(\lambda_1 - \lambda_2)^2 (u(x, z)v(x, z))^{\frac{m-n-\xi(m-n)}{2}}} \left[(\lambda_1 \lambda_2)^n [\lambda_2^{m-n+1} + \lambda_1^{m-n+1} - \lambda_1 \lambda_2^{m-n} - \lambda_2 \lambda_1^{m-n}] \right] \\ &= \frac{u(x, z)[u(x, z)v(x, z)]^{-n}}{(\lambda_1 - \lambda_2)^2 (u(x, z)v(x, z))^{\frac{m-n-\xi(m-n)}{2}}} \left[(-2h(z)u(x, z)v(x, z))^n ((\lambda_1)^{m-n}(\lambda_1 - \lambda_2) - (\lambda_2)^{m-n}(\lambda_1 - \lambda_2)) \right] \\ &= (-2h(z))^n u(x, z)j_{m-n}(x, y) \end{aligned}$$

Theorem 3.8 Sums involving Binomial Coefficients

For integers $n \geq 0$, prove that

- (i) $\sum_{i=0}^n \binom{n}{i} (2h(z))^{n-i} u(x, z)^{\xi(i)} [u(x, z)v(x, z)]^{\lfloor \frac{i}{2} \rfloor} j_i(x, z) = j_{2n}(x, z)$
- (ii) $\sum_{i=0}^n \binom{n}{i} (2h(z))^{n-i} u(x, z)^{\xi(i+1)} [u(x, z)v(x, z)]^{\lfloor \frac{i+1}{2} \rfloor} j_{i+1}(x, z) = j_{2n+1}(x, z)$
- (iii) $\sum_{i=0}^n \binom{n}{i} (2h(z))^{n-i} u(x, z)^{\xi(i)} [(u(x, z)v(x, z))]^{\lfloor \frac{i}{2} \rfloor} \hat{c}_{i+1}(x, z) = \hat{c}_{2n+1}(x, z)$
- (iv) $\sum_{i=0}^n \binom{n}{i} (2h(z))^{n-i} u(x, z)^{\xi(i+1)} [(u(x, z)v(x, z))]^{\lfloor \frac{i+1}{2} \rfloor} \hat{c}_i(x, z) = u(x, z) \hat{c}_{2n}(x, z)$

Proof: Proof of (i) part, by using equation (3.2.1) and binomial expansion we get

$$\begin{aligned} \sum_{i=0}^n \binom{n}{i} (2h(z))^{n-i} u(x, z)^{\xi(i)} (u(x, z)v(x, z))^{\lfloor \frac{i}{2} \rfloor} j_i(x, z) &= \sum_{i=0}^n \binom{n}{i} (2h(z))^{n-i} u(x, z) \left(\frac{\lambda_1^i - \lambda_2^i}{\lambda_1 - \lambda_2} \right) \\ &= u(x, z) \left(\frac{1}{\lambda_1 - \lambda_2} \right) \left[\sum_{i=0}^n \binom{n}{i} (2h(z))^{n-i} \lambda_1^i - \sum_{i=0}^n \binom{n}{i} (2h(z))^{n-i} \lambda_2^i \right] \\ &= u(x, z) \left(\frac{1}{\lambda_1 - \lambda_2} \right) [(\lambda_1 + 2h(z))^n - (\lambda_2 + 2h(z))^n] \\ &= u(x, z) \left(\frac{1}{\lambda_1 - \lambda_2} \right) \left[\frac{\lambda_1^{2n}}{(u(x, z)v(x, z))^n} - \frac{\lambda_2^{2n}}{(u(x, z)v(x, z))^n} \right] \\ &= u(x, z) \left(\frac{1}{\lambda_1 - \lambda_2} \right) \left[\frac{\lambda_1^{2n} - \lambda_2^{2n}}{(u(x, z)v(x, z))^n} \right] = j_{2n}(x, z) \end{aligned}$$

Similarly, we can prove (ii), (iii) and (iv).

Theorem 3.9: For every $m, n, r \in \mathbb{N}$ the relationship between bivariate bi-periodic Jacobsthal and bivariate bi-periodic Jacobsthal-Lucas polynomial sequences are given by

$$(i) \quad u(x, z)^{-\xi(n+r)} v(x, z)^{\xi(n+r)} \hat{c}_{n-r}(x, z) \hat{c}_{n+r}(x, z) - u(x, z)^{-\xi(n)} v(x, z)^{\xi(n)} \hat{c}_n^2(x, z) = (-2h(z))^{n-r} (\lambda_1 - \lambda_2)^2 (u(x, z)^{\xi(r)-2} v(x, z)^{-\xi(r)} j_r^2(x, z))$$

$$(ii) \quad u(x, z)^{\xi(mn+m)} v(x, z)^{\xi(mn+n)} \hat{c}_{m+1}(x, z) \hat{c}_n(x, z) - \\ u(x, z)^{\xi(mn+n)} v(x, z)^{\xi(mn+m)} \hat{c}_m(x, z) \hat{c}_{n+1}(x, z) = (-2h(z))^n [(u(x, z)v(x, z) + \\ 8h(z)] j_{m-n}(x, z)$$

Proof: Theorem can be proved easily by using Binet’s formula (3.2.1) and properties of parity function $\xi(n) = n - 2 \lfloor \frac{n}{2} \rfloor$.

Let us assume

$$u(x, z)^{-\xi(n+r)} v(x, z)^{\xi(n+r)} \hat{c}_{n-r}(x, z) \hat{c}_{n+r}(x, z) = H(x, z)$$

and

$$u(x, z)^{-\xi(n)} v(x, z)^{\xi(n)} \hat{c}_r^2(x, z) = F(x, z)$$

where

$$H(x, z) = \frac{[u(x, z)v(x, z)]^{-n}}{1} \left[\frac{\lambda_1^{2n} + (\lambda_1\lambda_2)^{n-r}(\lambda_2^{2r} + \lambda_1^{2r}) + \lambda_2^{2n}}{1} \right]$$

And

$$F(x, z) = \frac{[u(x, z)v(x, z)]^{-n}}{1} \left[\frac{\lambda_1^{2n} + (\lambda_1\lambda_2)^{2n} + \lambda_2^{2n}}{1} \right]$$

Now we subtract $H(x, z)$ and $F(x, z)$ we get,

$$H(x, z) - F(x, z) = (-2h(z))^{n-r} (\lambda_1 - \lambda_2)^2 (u(x, z)^{\xi(r)-2} v(x, z)^{-\xi(r)} j_r^2(x, z))$$

Similarly, we can solve (ii)

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