IJBPAS, February, 2016, 5(2), Special Issue: 116-122
ISSN: 2277–4998

International Journal of Biology, Pharmacy and Allied Sciences (IJBPAS)
'A Bridge Between Laboratory and Reader'

www.ijbpas.com

ASYMPTOTIC EIEGN VALUES OF-$W'' + q(x)\,W = \lambda\phi^2(x)\,W$ WITH THREE TURNING POINTS AND NUMANNCONDITIONS

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ABSTRACT

The present paper is concerned the second-order differential equation $-W'' + q(x)\,W = \lambda\phi^2(x)\,W$**(*)**with Neumann boundary conditions. By using the asymptotic solutions we find the distribution of eigen values of (*) in three turning points.

Keywords: Asymptotic eigenvalues, Neumann conditions, Turning points.

INTRODUCTION

Let us consider the boundary values problems of the form

$-W'' + q(x)\,W = \lambda\phi^2(x)\,W$ *for* $x \in I = [0, 1]$ (1)

where $\lambda = \rho^2$ is the spectral parameter; ϕ^2 and q are real functions. We suppose that

$$\phi^2(x) = \prod_{i=1}^{3}(x - x_i)^{l_i}\,\phi_0(x) \qquad (2)$$

where $0 < x_1 < x_2 < x_3 < 1$, $l_i \in N$, $\phi_0(x) > 0$ for $x \in I = [0, 1]$ and ϕ_0 is twice continuously differentiable function on I. On other words, ϕ^2 has in three zeros x_i, $i = 1, 2, 3$ of order l_i, the zeros x_i of ϕ^2 are called turning points. In this paper we obtained the asymptotic eigenvalues of equation (1) in three turning points case with Neumann conditions $W'(0) = W'(1) = 0$.

Notations

In the real second-order differential equation

$$-W'' + q(x)W = \lambda\phi^2(x)W \qquad for \qquad x \in I = [0, 1] \qquad (3)$$

$\phi^2(x)$ has in I, there zeros x_i of order l_i , $i=1, 2, 3$ where, l_1 is even, l_2 is odd and l_3 is even. Let $\varepsilon > 0$ be fixed sufficiently small and let

$$D_{i,\varepsilon} = [x_i + \varepsilon , x_{i+1} - \varepsilon] , \quad i = 1 , 2 \qquad D_{3,\varepsilon} = [x_3 + \varepsilon , 1] \tag{4}$$
$$I_{i,\varepsilon} = [x_{i-1} + \varepsilon , x_i - \varepsilon] \cup [x_i - \varepsilon , x_{i+\varepsilon}] \cup [x_{i+\varepsilon} , x_{i+1} - \varepsilon].$$

In [1] distinguished four different type of turning points: for $1 \le \nu \le m$

$$T_\nu = \begin{cases} I & \text{if } l_\nu \text{ is even and } \phi^2(x)(x-x_\nu)^{-l_\nu} < 0 \text{ in } I_{\nu\varepsilon}, \\ II & \text{if } l_\nu \text{ is even and } \phi^2(x)(x-x_\nu)^{-l_\nu} > 0 \text{ in } I_{\nu\varepsilon}, \\ III & \text{if } l_\nu \text{ is odd and } \phi^2(x)(x-x_\nu)^{l_\nu} < 0 \text{ in } I_{\nu\varepsilon}, \\ IV & \text{if } l_\nu \text{ is odd and } \phi^2(x)(x-x_\nu)^{l_\nu} > 0 \text{ in } I_{\nu\varepsilon}. \end{cases} \tag{5}$$

We know from [1] that x_1 is type of I, x_2 is type of IV and x_3 is of type II.

$\mu_i = \dfrac{1}{2+l_i}$, $\theta = \min\{\mu_1 , \mu_2 , \mu_3\}$ we know from [1] that the sectors S_{-1} is in form of

$S_{-1} = \{\rho \mid \arg\rho \in [-\frac{\pi}{4} , 0]\}$ we use for convenience the abbreviations

$[1] = 1 + O(\frac{1}{\rho^\theta}).$

The fundamental system solutions for $x \in I = [0 , 1]$

Now, let $W(x , \lambda)$ be the solution of equation (1). The fundamental system of solutions (FSS) for equation (1), when x_1 can be represented in the form (see [1] page 219)

since x_1 is type of I, we have the following FSS for $\rho \in S_{-1}$

If x_ν be a turning point of type I, then the estimates for $W^I_{\nu_1}(x , \rho)$, $W^I_{\nu_2}(x , \rho)$ are obtained from the corresponding estimates for $W^I_{\nu_1}(x , \rho)$, $W^I_{\nu_2}(x , \rho)$ by substituting there in ρ by $i\rho$. The FSS of (1) for x_1 that is type of I whit sector S_{-1} are the following form

In this section the FSS of (1) for sector $S_{-1} = \{\rho \mid \arg\rho \in [-\frac{\pi}{4} , 0]\}$ are the following form

$$W_{1,1}(x\,,\,\rho)=\begin{cases}|\phi(x)|^{-\frac{1}{2}}e^{\rho\int_{x_1}^{x}|\phi(t)|dt}[1] & 0\le x<x_1\\ |\phi(x)|^{-\frac{1}{2}}\csc\pi\,\mu_1 e^{\rho\int_{x_1}^{x}|\phi(t)|dt}[1] & x_1<x<x_2\end{cases} \quad (5)$$

$$W_{2,1}(x\,,\,\rho)=\begin{cases}|\phi(x)|^{-\frac{1}{2}}e^{-\rho\int_{x_1}^{x}|\phi(t)|dt}[1] & 0\le x<x_1\\ |\phi(x)|^{-\frac{1}{2}}\sin\pi\,\mu_1 e^{-\rho\int_{x_1}^{x}|\phi(t)|dt}[1] & x_1<x<x_2,\end{cases} \quad (6)$$

The *Wronskian* of FSS satisfies in following form $W(W_{1,1}(x\,,\,\rho)\,,W_{2,1}(x\,,\,\rho))=-2\rho[1]$.

Asymptotic form of the solutionsfor $W'(0)=W'(1)=0$

Let us consider the differential equation (1) with following boundary conditions

$C(0\,,\,\lambda)=1\,,\,C'(0\,,\,\lambda)=0.$ (7)

By applying the FSS $W_{1,1}(x\,,\,\rho)\,,W_{2,1}(x\,,\,\rho)$, for $x\in I_{1,\varepsilon}$ we have

$$C(x\,,\,\rho)=c_1W_{1,2}(x\,,\,\rho)+c_2W_{2,2}(x\,,\,\rho).$$

By derivation from $C(x\,,\,\rho)$ we can write

$$C'(x\,,\,\rho)=c_1W'_{1,1}(x\,,\,\rho)+c_2W'_{2,1}(x\,,\,\rho) \qquad \textit{for} \qquad x\in I_{1,\varepsilon}.$$

$$\begin{cases}C(x\,,\,\rho)=c_1W_{1,1}(x\,,\,\rho)+c_2W_{2,1}(x\,,\,\rho)\\ C'(x\,,\,\rho)=c_1W'_{1,1}(x\,,\,\rho)+c_2W'_{2,1}(x\,,\,\rho)\end{cases}\Rightarrow\begin{cases}C(0\,,\,\rho)=c_1W_{1,1}(0\,,\,\rho)+c_2W_{2,1}(0\,,\,\rho)=1\\ C'(0\,,\,\rho)=c_1W'_{1,1}(0\,,\,\rho)+c_2W'_{2,1}(0\,,\,\rho)=0\end{cases} \quad (8)$$

We infer by using Cramer's rule leads to the following equation

$$C(x\,,\,\rho)=\frac{1}{W(\rho)}\left(W_{1,1}(x\,,\,\rho)W'_{2,1}(0\,,\,\rho)-W'_{1,1}(0\,,\,\rho)W_{2,1}(x\,,\,\rho)\right) \quad (9)$$

Where, $W(\rho)=-2\rho[1]$.

Derivative of solutions and asymptotic eigenvalues

Let us consider boundary value problem $L_1=L_1(\phi^2(x)\,,\,q(x)\,,\,b)$ for equation (1) with boundary conditions

$C(0\,,\,\lambda)=1\,,\,C'(b,\lambda)=0\,,\,C'(0\,,\,\lambda)=0$ (10) .

The boundary value problem L_1 for $b\in(0,x_1)$ has a countable set of positive eigenvalues.

Now for fixed $x\in(0\,,\,x_1)$ and use (5),(6) we determine the connection coefficients $T_1(\rho)\,,\,T_2(\rho)$

$$C(x\,,\,\rho)=T_1(\rho)W_{1,1}+T_2(\rho)W_{2,1}\Rightarrow C'(x\,,\,\rho)=T_1(\rho)W'_{1,1}+T_2(\rho)W'_{2,1}. \quad (11)$$

$$For,.\ 0<x<x_1 \Rightarrow \begin{cases} W_{1,1}(x\ ,\ \rho)=|\phi(x)|^{-\frac{1}{2}}\left(e^{\rho\int_{x_1}^{x}|\phi(t)|dt}[1]\right) \\ W_{2,1}(x\ ,\ \rho)=|\phi(x)|^{-\frac{1}{2}}\left(e^{-\rho\int_{x_1}^{x}|\phi(t)|dt}[1]\right) \end{cases} \qquad (12)$$

Let consider, $A=e^{\rho\int_{x_2}^{x}|\phi(t)|dt}[1]+e^{-\rho\int_{x_2}^{x}|\phi(t)|dt}[1] \Rightarrow W_{1,1}(x\ ,\ \rho)=|\phi(x)|^{-\frac{1}{2}}A$

The derivative of $W_{1,1}(x\ ,\rho)$ is following form

$$W'_{1,1}(x\ ,\ \rho)=\left((|\phi(x)|^{-\frac{1}{2}})'A+|\phi(x)|^{-\frac{1}{2}}A'\right)$$

By use of fundamental theorem the derivative of A is as follow

$$\begin{cases} A'=\rho|\phi(x)|\left(e^{\rho\int_{x_1}^{x}|\phi(t)|dt}[1]-e^{-\rho\int_{x_2}^{x}|\phi(t)|dt}[1]\right)=\rho|\phi(x)|B \\ B=e^{\rho\int_{x_2}^{x}|\phi(t)|dt}[1]-e^{-\rho\int_{x_2}^{x}|\phi(t)|dt}[1] \end{cases} \qquad (13)$$

So $W'_{1,1}(x,\rho)$ is in following form

$$\begin{cases} W'_{1,1}(x,\rho)=\left((|\phi(x)|^{-\frac{1}{2}})'A+\rho|\phi(x)|B\right)=\rho\left(\frac{1}{\rho}(|\phi(x)|^{-\frac{1}{2}})'A+|\phi(x)|B\right), \\ C=(|\phi(x)|^{-\frac{1}{2}})', D=|\phi(x)| \Rightarrow W'_{1,1}(x,\rho)=\rho(\frac{1}{\rho}CA+DB) \Rightarrow \\ W'_{1,1}(x,\rho)=\rho DB(1+\frac{1}{\rho}\frac{CA}{DB}) \Rightarrow W'_{1,1}(x,\rho)=\rho DB\left(1+O(\frac{1}{\rho})\right) \end{cases} \qquad (14)$$

At last,

$$W'_{1,1}(x,\rho)=\rho DB\left(1+O(\frac{1}{\rho})\right)=\rho|\phi(x)|\left(e^{\rho\int_{x_2}^{x}|\phi(t)|dt}[1]-e^{-\rho\int_{x_2}^{x}|\phi(t)|dt}[1]\right)\left(1+O(\frac{1}{\rho})\right)$$

Similarly the derivative of $W_{2,1}(x\ ,\ \rho)=|\phi(x)|^{-\frac{1}{2}}\left(e^{-\rho\int_{x_1}^{x}|\phi(t)|dt}[1]\right)$ is as following form so,

Let us suppose $M=e^{-\rho\int_{x_2}^{x}|\phi(t)|dt}[1] \Rightarrow M'=-\rho|\phi(x)|e^{-\rho\int_{x_2}^{x}|\phi(t)|dt-i\frac{\pi}{4}}[1]=-\rho|\phi(x)|M$

$$W_{2,1}(x,\rho)=|\phi(x)|^{-\frac{1}{2}}M$$

$$\begin{cases} W'_{2,1}(x,\rho)=\left((|\phi(x)|^{-\frac{1}{2}})'M+|\phi(x)|^{-\frac{1}{2}}M'\right)=\left(KM-iM|\phi(x)|^{\frac{1}{2}}\right)=2\rho\sin\frac{\pi\mu_2}{2}\left(\frac{1}{\rho}KM-ML\right) \\ K=(|\phi(x)|^{-\frac{1}{2}})',L=-i|\phi(x)|^{\frac{1}{2}}\Rightarrow V'_{2,2}(x,\rho)=\rho(ML)\left(1-\frac{1}{\rho}(\frac{K}{L})\right)=\rho(ML)\left(1+O(\frac{1}{\rho})\right) \end{cases} \quad (15)$$

Finally $W'_{2,1}(x,\rho)$ is in form of

$$W'_{2,1}(x,\rho)=-\rho|\phi(x)|^{\frac{1}{2}}(e^{-\rho\int_{x_2}^{x}|\phi(t)|dt}[1])\left(1+O(\frac{1}{\rho})\right). \quad (16)$$

Hence we have estimated the solution of (1) defined by the initial condition (7) and Cramer's rule to determine the connection coefficients $T_1(\rho)$, $T_2(\rho)$ with

$$\begin{cases} C(x,\rho)=T_1(\rho)W_{1,1}(x,\rho)+T_2(\rho)W_{2,1}(x,\rho) \\ C'(x,\rho)=T_1(\rho)W'_{1,1}(x,\rho)+T_2(\rho)W'_{2,1}(x,\rho) \end{cases}\Rightarrow\begin{cases} T_1(\rho)W_{1,1}(0,\rho)+T_2(\rho)W_{2,1}(0,\rho)=1 \\ T_1(\rho)W'_{1,1}(0,\rho)+T_2(\rho)W'_{2,1}(0,\rho)=0 \end{cases} \quad (17)$$

$$\begin{cases} T_1(\rho)=\dfrac{W'_{2,1}(0,\rho)}{-2\rho[1]}=-|\phi(0)|^{\frac{1}{2}}(e^{-\rho\int_{x_2}^{0}|\phi(t)|dt})\left(1+O(\frac{1}{\rho})\right) \quad (18) \\ T_2(\rho)=\dfrac{W'_{1,1}(0,\rho)}{2\rho[1]}=|\phi(0)|^{\frac{1}{2}}\left(e^{\rho\int_{x_2}^{0}|\phi(t)|dt}-e^{-\rho\int_{x_2}^{0}|\phi(t)|dt}\right)\left(1+O(\frac{1}{\rho})\right) \end{cases}$$

$$T_2(\rho)=|\phi(0)|^{\frac{1}{2}}\Gamma_1\ ,T_1(\rho)=-|\phi(0)|^{\frac{1}{2}}\Gamma_2 \quad (19)$$

$$\begin{cases} \Gamma_1=\left(e^{\rho\int_{x_1}^{0}|\phi(t)|dt}-e^{-\rho\int_{x_1}^{0}|\phi(t)|dt}\right),\ \Gamma_4=(e^{-\rho\int_{x_1}^{x}|\phi(t)|dt-i\frac{\pi}{4}}[1])\left(1+O(\frac{1}{\rho})\right) \\ \Gamma_3=\left(e^{\rho\int_{x_1}^{x}|\phi(t)|dt}[1]-e^{-\rho\int_{x_1}^{x}|\phi(t)|dt}[1]\right)\left(1+O(\frac{1}{\rho})\right),\ \Gamma_2=(e^{-\rho\int_{x_1}^{0}|\phi(t)|dt}) \\ V'_{2,2}(x,\rho)=-\rho|\phi(x)|^{\frac{1}{2}}\Gamma_4 \qquad ,\dot{W}'_{1,1}(x,\rho)=\rho|\phi(x)|^{\frac{1}{2}}\Gamma_3 \quad (20) \end{cases}$$

By substituting (20)and (19) in (18) we obtain the leading term as $C'(x,\rho)$ follows

$$\begin{cases} C'(x,\rho)=-|\phi(0)|^{\frac{1}{2}}\Gamma_1\rho|\phi(x)|^{\frac{1}{2}}\Gamma_4-|\phi(0)|^{\frac{1}{2}}\Gamma_2\rho|\phi(x)|^{\frac{1}{2}}\Gamma=|\phi(0)|^{\frac{1}{2}}\Gamma_2\rho|\phi(x)|^{\frac{1}{2}}\Gamma_3+\Gamma_1\rho|\phi(x)|^{\frac{1}{2}}\Gamma \\ C'(x,\rho)=|\phi(0)|^{\frac{1}{2}}\rho(\Gamma_2\Gamma_3+\Gamma_1\Gamma_4) \quad (21). \end{cases}$$

Now must to determine the value of $\Gamma_2\Gamma_3\ and\ \Gamma_1\Gamma_4$

$$\begin{cases} \Gamma_2\Gamma_3=(e^{-\rho\int_{x_1}^{0}|\phi(t)|dt})\left(e^{\rho\int_{x_1}^{x}|\phi(t)|dt}[1]-e^{-\rho\int_{x_1}^{x}|\phi(t)|dt}[1]\right)\left(1+O(\frac{1}{\rho})\right), \\ \Gamma_1\Gamma_4=(e^{-\rho\int_{x_1}^{x}|\phi(t)|dt}[1])\left(e^{\rho\int_{x_1}^{0}|\phi(t)|dt}-e^{-\rho\int_{x_1}^{0}|\phi(t)|dt+}\right)\left(1+O(\frac{1}{\rho})\right) \quad (22) \end{cases}$$

By applying $C'(x\ ,\ \rho)$ =0, consequently $\Gamma_2\Gamma_3+\Gamma_1\Gamma_4$ =0

$$\begin{cases}\Gamma_2\Gamma_3=-\Gamma_1\Gamma_4\Rightarrow(e^{-\rho\int_{x_1}^{0}|\phi(t)|dt})\left(e^{\rho\int_{x_1}^{x}|\phi(t)|dt}[1]-e^{-\rho\int_{x1}^{x}|\phi(t)|dt}[1]\right)=\\(e^{-\rho\int_{x1}^{x}|\phi(t)|dt}[1])\left(e^{-\rho\int_{x1}^{0}|\phi(t)|dt}-e^{\rho\int_{x1}^{0}|\phi(t)|dt}\right)\end{cases}$$

$$\begin{cases}[1](e^{\rho\int_{x_1}^{x}|\phi(t)|dt}-e^{-\rho\int_{x1}^{x}|\phi(t)|dt})(e^{\rho\int_{x1}^{x}|\phi(t)|dt})=\\(e^{\rho\int_{x1}^{0}|\phi(t)|dt})(e^{-\rho\int_{x1}^{0}|\phi(t)|dt}-e^{\rho\int_{x1}^{0}|\phi(t)|dt})\\ [1](e^{2\rho\int_{x1}^{x}|\phi(t)|dt}-1)=(1-e^{2\rho\int_{x_1}^{0}|\phi(t)|dt})\end{cases}$$

$$\begin{cases}[1]=\dfrac{1-e^{2\rho\int_{x1}^{0}|\phi(t)|dt}}{e^{2\rho\int_{x1}^{x}|\phi(t)|dt}-1}=1+O(\dfrac{1}{\rho})\Rightarrow K=1-e^{2\rho\int_{x_1}^{0}|\phi(t)|dt}\\ \dfrac{K}{e^{2\rho\int_{x1}^{x}|\phi(t)|dt}-1}=1+O(\dfrac{1}{\rho})\Rightarrow\dfrac{K}{1+O(\dfrac{1}{\rho})}=e^{2\rho\int_{x_2}^{x}|\phi(t)|dt-i\frac{\pi}{4}}-1\\ K+O(\dfrac{1}{\rho})=e^{2i\rho\int_{x_2}^{x}|\phi(t)|dt-i\frac{\pi}{4}}-1\Rightarrow K+1+O(\dfrac{1}{\rho})=e^{2\rho\int_{x1}^{x}|\phi(t)|dt}\end{cases}\tag{23}$$

$$Ln\left(K+1+O(\frac{1}{\rho})\right)=O(\frac{1}{\rho^{\theta}})=2\rho\int_{x_1}^{x}|\phi(t)|dt-\frac{\pi}{4}-2k\pi i\tag{24}$$

By dividing (24) to the $2\int_{x1}^{x}|\phi(t)|dt$ we obtain the leading term ρ as follows

$$\rho_k=\frac{\frac{\pi}{4}+2k\pi}{2\int_{x1}^{x}|\phi(t)|dt}+O(\frac{1}{\rho^{\theta}})=\frac{k\pi+\frac{\pi}{8}}{\int_{x1}^{x}|\phi(t)|dt}+O(\frac{1}{\rho^{\theta}})$$

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